Homework 8

Mathematics in Computer Science

1. Prove that the smallest group containing I, r, and f is $G = \{I, r, r^2, \ldots, fr^3\}$.

2. What is the group $\{I, a, b, c, \ldots\}$ with $ac = ca, bc = cb, aa = I, bb = I, cc = I, bab = I$.

3. (a) What is the set of physical symmetries of the cube?

   (b) Give a brief discussion that will convince us that your answer to part (a) is correct.

4. (a) What are the symmetries of the tic tac toe board game?

   (b) Construct the group multiplication table for the symmetries.

5. Consider tic tac toe boards that are completely filled in with $\bigcirc$'s and $\times$'s. Use Burnside's theorem to determine the number of equivalence classes.

6. Let $b_1, b_2, \ldots$ be the various board positions of tic tac toe. Let G be a group of transformations that map board positions to equivalent board positions.

   (a) How many board configurations are there if each position on the board contains an $\bigcirc$, a $\times$, or is open?

   (b) Prove that the number of elements of G that preserve a board configuration b_1 is the same as the number of group elements that map b_1 to an equivalent configuration b_2.

   (c) How many elements are there in the equivalence class of b_1 if $|G| = m$ and k elements of G preserve b_1.

7. Consider two operations on a 2×2 square. The operations are: r which is a rotation of $180°$, and f which is a flip about the diagonal axes from upper left to lower right.

   (a) What is minimum number of additional operations that needed to be added to form a group?

   (b) Write out the group table.

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \xrightarrow{r} \begin{array}{|c|c|}\hline 4 & 3 \\ \hline 2 & 1 \\ \hline \end{array}$$

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \xrightarrow{f} \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$$