Homework 2

Mathematics in Computer Science

1. If there is a one-to-one mapping from A to B and also an onto mapping from A to B does this imply a one-to-one onto mapping?

2. Sketch an algorithm to list all triples of a countably infinite set.

3. A rational is a number that is the ratio of two integers.

   (a) Prove every rational is a terminating real or a repeating real

   (b) Prove the converse. Hint: Prove that $\{0.1.^i10^i10^i \cdots | i \in Z\}$ is rational and if $a = bc$ and two of a, b, c are rational then the third is rational.

4. Prove that an integer is either a perfect square or its square root is irrational.

5. Prove that

   (a) between every two rationals there is a real;

   (b) between every two reals there is a rational.

6. Are the sets of all points on each of two lines the same cardinality even if the lines are different lengths?

7. **(Halting problem)** Suppose you could write a computer program that would take as input another computer program and an input to it. Your computer program would halt on all inputs and correctly state whether the program that was inputed would halt. What contradiction would this lead to? Hint: Could you compute something that is not computable?

8. What if I diagonalize over all rationals and claim I have a rational that is not on the list of all rationals? Explain what is wrong with the reasoning.

9. What is wrong with the following proof that leads to the erroneous conclusion that the rationals between 0 and 1 are not countable? Convert each rational to decimal notation. If the rational is a terminating decimal extend the decimal by adding zeros so all the rationals are infinite length decimals. Assume that the set of rationals is countably infinite. Then create a list of all rationals. The rational that differs from each rational in the list by differing from the i^{th} rational in the i^{th} digit is not in the list, a contradiction. Thus, the rational are not countably infinite.

Think about the following questions (Optional questions, no need for submission)

1. Is there a one-to-one mapping from the reals R to R^2?

2. Is there a one-to-one onto mapping from $[0, 1) \leftrightarrow [0, 1) \times [0, 1)$?

3. Let

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational} \end{cases}$$

and

$$g(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

What do you think the value of the two integrals

$$\int_0^1 f(x)dx$$

$$\int_0^1 g(x)dx$$

will be? What is the function $f(n) + g(n)$ and what is the value of the integral $\int_0^1 (f(x) + g(x))dx$? Credit will be given on how thoughtful your answers are and how much they show your understanding of the concepts of countably and not countably infinite.