Functional Analysis

Exercices III- Space of continuous functions

1. Check if the following subsets of $C^0([-1,1],\mathbb{R})$ are equicontinuous or not at 0 :

$$\{x \to \frac{1}{n}\cos(nx) + (1+x)^\alpha,\ n = 1,2,...,\ \alpha \in [-1,1]\}$$

$$\{x \to \exp(\lambda + x),\ \lambda \in \mathbb{R}\}$$

$$\{x \to \arctan(nx),\ n \in \mathbb{N}\}$$

2. For the following two sequences f_n ($n \geq 1$), find their limit and check the uniform convergence on $[0,1]$.
   (a) $f_n(x) = x^n$
   (b) $f_n(x) = (1-x)x^n$.
   Let now f_n be a sequence of real functions, which are non decreasing over $[a,b]$. We assume a pointwise convergence to a continuous function g over $[a,b]$.
   (c) Show that the convergence is uniform.
3. Let E a metric space and F a Banach space. We assume that a sequence f_n of $C^0_b(E,F)$ converges uniformly over a dense subset D of E. Show that the convergence is uniform on the whole of E.