IV- HILBERT SPACES

1. Inner products

In view of many applications, it is necessary to distinguish between real and complex vector spaces.

Definition 1.1. *Let E be a* **real** *vector space. An inner product on E is a real function defined on $E \times E$, denoted by $(.,.)$ which is linear w.r.t. the first argument, linear w.r.t. the second argument, symmetric, and:*
a) $(x, x) \geq 0$ for all $x \in E$
b) $(x, x) =$ iff $x = 0$

This is equivalent to saying that this is a real symmetric sesquilinear form, which is definite positive. This definition gives an extension of the usual euclidean inner product on $\mathbb{R}^N$. When extending to complex vector spaces, we must introduce the conjuguate.

Definition 1.2. *Let E be a* **complex** *vector space. An inner product on E is a complex function, defined on $E \times E$, linear w.r.t. the first argument, anti linear w.r.t. the second argument, and definitive positive and hermitian.*

Thus we have

$$(x, x) \geq 0 \text{ for all } x \in E \text{ and } (x, x) = 0 \text{ iff } x = 0.$$

Moreover

$$(x, y) = \overline{(y, x)}.$$

For example, on $\mathbb{C}^N$, $(x, y) \equiv \sum_{k=1}^{N} x_k \bar{y}_k$ is the standard inner product.
Sometimes, the linearity and the anti linearity are exchanged.
In any case, a real or complex vector space endowed with an inner product is called an inner product space.

Remark 1.3. *If (X, Σ, μ) is a measure space, then consider the space $E = L^2_{\mathbb{C}}(X)$ and the function $(.,.) : E \times E \to \mathbb{C}$, $(f, g) = \int_X f \bar{g} d\mu$. Then this is the standard inner product. Similarly, we can consider the space l^2.*
It is also clear that a subspace of an inner product space is also an inner product space. Furthermore, the product of two inner product spaces can be made an inner product space.

The following result shows that an inner product space is in fact a vector normed space, and also gives the corresponding Cauchy Schwarz inequality:

Proposition 1.4. *Let E be an inner product space. Then*
$|(x, y)| \leq (x, x)(y, y)$ for all $x, y \in E$.
$||x|| \equiv (x, x)^{\frac{1}{2}}$ defines a norm.

Proof. We just prove the inequality. First note that for any scalar α and β, for any $x, y \in E$, one has

$$0 \leq (\alpha x + \beta y, \alpha x + \beta y) = |\alpha|^2 (x, x) + \alpha \bar{\beta} (x, y) + \beta \bar{\alpha} (y, x) + |\beta|^2 (y, y)$$

Choose

$$\alpha = -\overline{(x, y)}/(x, x) \text{ and } \beta = 1$$

(assuming $x \neq 0$ and $y \neq 0$).

We obtain immediately the inequality. For the other part, be careful noticing

$$||x + y||^2 = ||x||^2 + 2Re\ (x, y) + ||y||^2$$

□

The norm thus obtained is called the induced norm. We have also $|(x, y)| \leq ||x||\ ||y||$.
For later use, we record the following properties

Lemma 1.5. *Let E be an inner product space. Then, for all x, y, u, v in E:*
a) $(u + v, x + y) - (u - v, x - y) = 2(u, y) + 2(v, x)$
b) (if E is complex) $4(u, y) = (u + v, x + y) - (u - v, x - y) + i(u + iv, x + iy) - i(u - iv, x - iy)$
c) (parallelogram rule) $||x + y||^2 + ||x - y||^2 = 2(||x||^2 + ||y||^2$
d) (if E is real) polarization identity $4(x, y) = ||x + y||^2 - ||x - y||^2$
e) (if E is complex) polarization indentity $4(x, y) = ||x + y||^2 - ||x - y||^2 + i||x + iy||^2 - i||x - iy||^2$

We have seen above that an inner product space is a normed vector space. The converse is not necessarily true. For example, if one wants to show that a given norm is not induced by an inner product, it suffices to show for example that it does not satisfy the parallelogram rule.

Remark 1.6. *In $C([0, 1])$, the standard norm is not induced by an inner product. Consider for example the functions 1 and x.*

In the following, we shall use the topology induced by the norm defined through the inner product.
One can show for example that in an inner product space E, if x_n and y_n are two convergent sequences, then (x_n, y_n) converges to (x, y).

2. Orthogonality

As usual, in an inner product space, x and y in E are said to be orthogonal if $(x, y) = 0$.
A set of vectors e_j, j in some set finite J, is said to be orthonormal if the vectors have all length one and are mutually orthogonal. It is not difficult to show that such a set is linearly independent.
Let us recall the Gram Schmidt algorithm. It gives one way to an explicit construction of this fact: let $\{v_1, ..., v_k\}$ be a linearly independent subset of an inner product space E. Let $F = Vect\{v_1, .., v_k\}$. Then there exists an orthonormal basis $\{e_1, ..., e_k\}$ for F.
This is proved by induction on k.
For $k = 1$, since $v_1 \neq 0$, we can define $e_1 = v_1/||v_1||$.
Assume next that the result is true for an arbitrary integer $k \geq 1$.
Let $\{v_1, ..., v_{k+1}\}$ be a linearly independent set. Let $\{e_1, ..., e_k\}$ be an orthonormal basis for $Vec\{v_1, ..., v_k\}$.
Of course v_{k+1} is not in $Vec\{e_1, ..., e_k\} = Vec\{v_1, ..., v_k\}$. Define

$$b_{k+1} = v_{k+1} - \sum_{n=1}^{k} (v_{k+1}, e_n)e_n$$

One can show that $b_{k+1} \in Vec\{v_1, ..., v_{k+1}\}$, that $b_{k+1} \neq 0$ and that b_{k+1} is orthogonal to all e_1, ...,e_k. Then set $e_{k+1} = b_{k+1}/||b_{k+1}||$.
In general, if $\{e_i\}$ is an orthonormal set, one can see that

$$|| \sum_i \alpha_i e_i ||^2 = \sum_i |\alpha_i|^2$$

for any $\alpha_i \in \mathbb{K}$.

Definition 2.1. *An inner product space which is complete for the natural topology is called an Hilbert space.*

For example, any finite dimensional inner product space, $L^2(X)$, l^2 are all examples of Hilbert spaces.

One can show that if E is an Hilbet space, and if F is a linear subspace of E, then F is an Hilbert space iff it is closed.

Definition 2.2. *Let E be an inner product space. Let F be a subset of E. The orthogonal complement of F is the set denoted by $F^\perp$ of all vectors x which are orthogonal to every vector of F.*

If $F = \emptyset$, $F^\perp = E$.

One has the following properties:

a) $0 \in F^\perp$

b) if $0 \in F$, then $F \cap F^\perp = \{0\}$; otherwise $F \cap F^\perp = \emptyset$.

c) $\{0\}^\perp = E$ and $E^\perp = \{0\}$.

d) If F contains an open ball, then $F^\perp = \{0\}$. Thus the orthogonal complement of any non empty set is always the neutral element.

e) If $F \subset G$, then $G^\perp \subset F^\perp$.

f) $F^\perp$ is a closed linear subspace of E (for any subset F).

g) $F \subset (F^\perp)^\perp$.

In the case where F is a linear subspace, we have

Lemma 2.3. *Let F be a linear subspace of an inner product space E. Then*

$$x \in F^\perp \textit{ iff } \|x - y\| \geq \|x\|, \forall y \in F$$

Proof. For the implication, expand $\|x - y\|^2$. For the opposite, replace y by αy and conclude. □

We can use this to show

Theorem 2.4. *Let A be a non empty closed and convex subset of a Hilbert space E. Let $p \in E$. Then there exists a unique $q \in A$ such that*

$$\|p - q\| =\in \{\|p - a\|, a \in A\}$$

Proof. Set γ for the infimum, which is well defined because the set is non empty and bounded from below. Let us show that q exists.

By definition of γ, for all $n \in \mathbb{N}$, there exists q_n in A such that

$$\gamma^2 \leq \|p - q_n\|^2 \leq \gamma^2 + 1/n$$

We show that this is a Cauchy sequence. Use the parallelogram rule to get

$$\|(p - q_n) + (p - q_m)\|^2 + \|(p - q_n) - (p - q_m)\|^2 = 2\|p - q_n\|^2 + 2\|p - q_m\|^2$$

Thus

$$\|2p - (q_n + q_m)|^2 + \|q_n - q_m\|^2 < 4\gamma^2 + 2(1/n + 1/m)$$

Since A is convex, then $(1/2)(q_m + q_n)$ is in A. Thus

$$\|2p - (q_n + q_m)\|^2 = 4\|p - \frac{1}{2}(q_m + q_n)\|^2 \geq 4\gamma^2$$

using the definition of γ and so

$$\|q_n - q_m\|^2 < 4\gamma^2 + 2(1/n + 1/m) - 4\gamma^2 = 2(1/n + 1/m)$$

Thus q_n is a Cauchy sequence and converges to some $q \in E$, because E is a Hilbert space. Since A is closed, $q \in A$ also. Moreover we obtain immediately $\|p - q\| = \gamma$.

For the uniqueness, suppose there exists another $w \in A$ such that $\|p - w\| = \gamma$. Then $\frac{1}{2}(q + w) \in A$ and thus $\|p - \frac{1}{2}(q + w)\| \geq \gamma$. From the parallelogram rule, we obtain

$$\|(p - w) + (p - q)\|^2 + \|(p - w) - (p - q)\|^2 = 2\|p - w\|^2 + 2\|p - q\|^2$$

and thus

$$\|q - w\|^2 = 2\gamma^2 + 2\gamma^2 - 4\|p - \frac{1}{2}(q + w)\|^2 \leq 4\gamma^2 - 4\gamma^2 = 0$$

thus $w = q$.

□

To remember: If A is a non empty closed and convex subset of a Hilbert space E, if p is a fixed point of E, therefore there exists a unique point in A which the closest point in A to p. In finite dimensions, if we do not assume that the set A is convex, we still have the existence of such a point q. However it might be non unique.

Remark 2.5. *In fact we can also give a similar result but without assuming that E is a Hilbert space. More precisely, if E is a inner product space, if A is a convex complete subset of E, then we have the same conclusion.*

For $x \in E$, the unique point as above in A is called the orthogonal projection on A and denoted by $P_A x$. One can show that P_A is a contraction.

Theorem 2.6. *Let F be a closed linear subspace of a Hilbert space E. For any $x \in E$, there exists a unique $y \in F$ and $z \in F^\perp$ such that $x = y + z$. Furthermore $\|x\|^2 = \|y\|^2 + \|z\|^2$.*

Proof. Since F is a non empty closed convex set, from the previous result, it follows that exists $y \in F$ such that for all $u \in F$, we have $\|x - y\| \leq \|x - u\|$. Then set $z = x - y$. Then for all $u \in F$, we have

$$\|z - u\| \geq \|x - (y + u)\| \geq \|x - y\| = \|z\|$$

and thus $z \in F^\perp$. Thus the couple (y, z) exists. The uniqueness can be proved easily, as well as the last equality.

□

Having in mind the above decomposition and notations, if F is a closed linear subspace of a Hilbert space E, the decomposition $x = y + z$ with $y \in F$ and $z \in F^\perp$ is called the orthogonal decomposition of x w.r.t. F.

Show the following facts:

a) If F is a closed linear subspace of a Hilbert space E, then $F^{\perp\perp} = F$.

b) If F is any linear subspace of a Hilbert space E, then $F^{\perp\perp} = \overline{F}$.

Let now E an inner product space. A sequence e_n in E is an orthonormal sequence if they have all unit norms and are mutually orthogonal.

For example in l^2 it is easy to construct an orthonormal basis. Also in $L^2_{\mathbb{C}}[-\pi, \pi]$, the functions $e_n(x) = (2\pi)^{-1/2} e^{inx}$ is an orthonormal sequence.

It is not difficult to show also that any infinite dimensional inner product space E contains an orthonormal sequence.

We have

Lemma 2.7. *(Bessel) Let E be an inner product space and e_n an orthonormal sequence in E. Then, for any $x \in E$, the series $\sum_n |(x, e_n)|^2$ converges and we have*

$$\sum_n \|(x, e_n)^2 \leq \|x\|^2$$

Proof. Set $y_k = \sum_{n=1}^k (x, e_n) e_n$. Then expand $\|x - y_k\|^2$ and conclude.

□

Lemma 2.8. *Let E be a Hilbert space and let e_n an orthonormal sequence in E. For any sequence of scalars α_n, the series $\sum_n \alpha_n e_n$ converges in E iff the series $\sum_n |\alpha_n|^2$ converges, that is $\alpha_n \in l^2$. In this case, we have*

$$\|\sum_n \alpha_n e_n\|^2 = \sum_n |\alpha_n|^2$$

Proof. **Do it** □

We deduce immediately that

Lemma 2.9. *Let E be a Hilbert space and e_n an orthonormal sequence in E. For any $x \in E$, the series $\sum_n (x, e_n)e_n$ converges in E.*

We expect that the previous series will converge to x. But that might be false. We have

Proposition 2.10. *Let E be a Hilbert space and let e_n be an orthonormal sequence in E. The following conditions are equivalent:*
a) $\{e_n, n \in \mathbb{N}\}^{\perp} = \{0\}$.
b) $\overline{Vec}\{e_n, n \in \mathbb{N}\} = E$.
c) $\|x\|^2 = \sum_{|}(x, e_n)|^2$ for all $x \in E$.
d) $x = \sum_n (x, e_n)e_n$ for all $x \in E$.

Proof. **Do it** □

This last result explains the utility of

Definition 2.11. *Let E be a Hilbert space and e_n an orthonormal sequence in E, We say that it is an orthonormal basis of E if any of the conditions of the previous result hods true.*

We have seen that any infinite dimensional Hilbert space contains an orthonormal sequence. Is it true that we can have the existence of an orthonormal basis. It appears that E must not be too big: it must be separable.

Theorem 2.12. *Any finite dimensional normed space is separable. Any infinite dimensional Hilbert space is separable iff it has an orthonormal basis.*

Proof. **Check this. Use the density of rational numbers**. □

For example, the Hilbert space l^2 is separable. Also $L^[0, \pi]$ is separable (fourier series).

3. Representation results

We shall assume in this section that E is a Hilbert space. Let E' his dual (the set of all linear and continuous forms on E).

Theorem 3.1. *(Riesz) Let $l \in E'$. Then there exists a unique $u \in E$ such that*

$$llv \in E,\ l(v) = (u, v)$$

Proof. **Do it** □

Using this result, one can show that if a is a continuous sesquilinear form on $E\times$, then there exists $a \in L(E)$ such that

$$\forall (u, v) \in E \times E,\ a(u, v) = (Au, v)$$

Note that a is continuous iff it is bounded: $|a(u, v)| \le C\|u\|\,\|v\|$. In this case $\|A\| \le C$.

One can use this result to show:

Proposition 3.2. *Let $A \in L(E)$. Then there exists a unique A^* in $L(E)$ suc that*

$$\forall (u, v) \in E \times E,\ (Au, v) = (u, A^* v)$$

It is called the adjoint of A. Moreover $\|A^*\| = \|A\|$ and $A^{**} = A$.

The proof is left as an **exercice**.

An operator $A \in L(E)$ is said to be hermitian iff $A = A^*$ (symmetric). For a given $A \in L(E)$, we can associate a natural sesquilinear form a. Then A is hermitian iff a is.

Theorem 3.3. *(Lax Milgram) Let a be a continuous sesquilinear form. Assume that it is coercive, that is there exists $\alpha > 0$ such that*

$$llu \in E,\ Re\ a(u, u) \geq \alpha \|u\|^2$$

Then for any linear (continuous) form $l \in E'$, the exists a unique vector $u \in E$ such that

$$llv \in E,\ a(u, v) = l(v)$$

For the proof, using Riez Lemma, we can associate a vector $f \in E$ to the form l and also an operator $a \in L(E)$ to the sesquilinear form a. Then we need to solve $Au = f$. Fix $t > 0$. Then this is equivalent to $u = F_t(u)$ with $F_t(u) = u - tAu + tf$.

Note that $(Au, u) \geq \alpha \|u\|^2$. Morover, one can show that we can choose t such that F_t satisfies Picard fixed point Theorem.