Functional Analysis

Exercices II- Normed vector spaces and Banach spaces

1. Let $\mathcal{P}$ denote the infinite dimensional vector space of polynomials on $[0,1]$, considered as a subspace of $C([0,1])$ with the L^∞ norm or with the L^1 norm. Show that these two norms are not equivalent on $\mathcal{P}$.
2. Let $S = \left\{x_n \in l^2,\ s.t.\ \exists N \in \mathbb{N} \text{ with } x_n = 0 \text{ for } n \geq N\right\}$. Show that S is not closed.
3. Let E be a normed space, $x \in E$, $x \neq 0$, and let F be a linear subspace of E.
   (a) If there exists $\eta > 0$ such that $B(0;\eta) \subset F$, show that $\frac{\eta x}{2\|x\|} \in F$.
   (b) If F is open, show that $F = X$.
4. Let $E = C^0([a,b],\mathbb{C})$ that we consider together with one of the norms $\|.\|_1$ or $\|.\|_\infty$. Check that E such equipped is a normed vector space. Show that $B^\infty(0;1) \subset B^1(0;1)$. Does there exists a $R > 0$ such that $B^1(0;1) \subset B^\infty(0;1)$? Is $B^\infty(0;1)$ closed for the norm $\|.\|_1$? Is $B^1(0;1)$ closed for the norm $\|.\|_\infty$?
5. Consider $C^0([-1,+1];\mathbb{C})$. Let $H : [-1,+1] \to \mathbb{C}$ such that $H(t) = -1$ for negative values, $+1$ for positive values and 0 for 0. We set
$$E = \{f = u + \lambda H, u \in C^0([-1,+1];\mathbb{C}),\ \lambda \in \mathbb{C}\}$$
   and for $f \in E$, we define $\|f\|_1 = \int_0^1 |f(t)|dt$.
   Check that E with this $\|.\|_1$ is a normed vector space.
   Show that the sequence $u_n(t) = \frac{2}{\pi}\arctan(nt)$ is a sequence of E which converges to H.
   For $f \in E$, show that there exists an unique $\lambda \in \mathbb{C}$ such that $f - \lambda H \in C^0([-1,+1];\mathbb{C})$. Set $l(f) = \lambda$.
   Check that l is linear but that it is not continuous.
6. Let $E = C([0,1])$ (complex valued functions) with the L^1 norm. Let $\Phi : E \to E$ defined by $\Phi(u)(t) = tu(t)$. Show that $\Phi \in L_c(E;E)$ with a norm less than 1. Using the sequence $u_n(t) = (n+1)t^n$, show that the norm of Φ is indeed 1. Show that there does not exist $u \neq 0$ such that $\|\Phi(u)\|_1 = \|u\|_1$.
7. Let V be a linear vector subspace of a vector space E of co dimension n. Show that there exists n independent linear forms $l_1, ... l_n$ such that $v \in V$ iff $v \in \cap_i Kerl_i$. Show that the l_i are continuous iff V is closed.
8. Let E be the space of sequences of complex numbers $a = (a_n)$ which are absolutely convergent (as a series). We set $\|a\| = \sum_n |a_n|$.
   (a) Show that $(E, \|.\|)$ is a normed vector space.
   (b) Let $e^{(k)} \in E$ the sequence defined by $e_n^{(k)} = 0$ for $n \neq k$ and $e_k^{(k)} = 1$. Has this sequence any closure point ? What can we deduce ?
   (c) For fixed $R > 0$, we consider $K = \left\{a \in E, s.t.\ |a_n| \leq Rn^{-n^2}\right\}$. Show that K is precompact.
   One can start by showing that the sets $K_N = \{a \in K, s.t.\ a_n = \ for\ n \geq N+1\}$ are precompact.

9. Let E be as in the preceding exercice and let u be a bounded non null sequence. For $a \in E$, we denote $l(a) = \sum_n u_n a_n$.
   (a) Show that l is a continuous linear form. Let $H = \ Ker\ l$ be its kernel and $v_n = \bar{u}_n / n^2$.
   (b) Show that any sequence $a \in E$ can be written as $a = \lambda v + b$, with $\lambda \in \mathbb{C}$ and $b \in H$ and that this decomposition is unique.
   Let $W = \{a \in E,\ s.t.\ \sum_n n a_n$ is abs. conv.$\}$. For $a \in E$, let $a^{(k)} = (a_1, a_2, ..., a_k, 0, ...)$.
   (c) Show that $a^{(k)} \to a$ in E. Is W a closed linear vector subspace of E ? Let $b \in E - W$ and $E_1 = W \oplus \ Vect(b)$. For $a = w + \lambda b$ in E_1, we define l as $l(a) = \lambda$. Show that l is a linear form on E_1 which is not continuous.
10. Let $E = L^1(\mathbb{R})$ with the usual norm. Show that
$$H = \{f \in E, s.t. \int_{-\infty}^{\infty} f(x)dx = \int_0^{\infty} (\frac{x}{1+x} f(x)dx\}$$
is a closed hyperplane and find a supplementary subspace.
11. Let E be a normed vector space over $\mathbb{K}$. Let $H = l^{-1}(0)$, where l is a continuous linear form. Show that $d(x, H) = \frac{|l(x)|}{\|l\|}$.
12. Let E be a normed vector space over $\mathbb{K}$. An affine hyperplane is by definition of the form $H = a + H_0$ where $a \in E$ and H_0 is an hyperplane. Show that there exists a linear form l and a scalar α such that $H = l^{-1}(\alpha)$.
13. We let L_0 the vector space over $\mathbb{R}$ of the sequences a of complex numbers converging to 0. For $a \in L_0$, let $\|a\| = \sup_n |a_n|$ and $l(a) = \sum_n \frac{a_n}{2^n}$.
   (a) Show that for all $a \in L_0$, we have $\|a\| < \infty$.
   (b) Show that l is a linear continuous form of norm 1 but that this norm is never obtained.
14. Let $E = C^0([-1, 1])$ (valued in $\mathbb{C}$), and we consider the usual $\|.\|_1$ norm. Show that the sequence u_n defined by $u_n(t) = n \arctan(nt)$ has a limit for the pointwise convergence denoted by u.
   Do we have $u \in E$? Show that u_n converges to u for the $\|.\|_1$ norm, and thus that it is a Cauchy sequence in E. Conclude. What about the $\|.\|_\infty$ norm ?
15. Determine if the following spaces are Banach :
   (a) $E = C^0([0, 1])$ with the L^2 norm.
   (b) $E = C^0_0(\mathbb{R}^N)$ with the L^∞ norm.
   (c) E is the space of sequences converging to 0 for the sup norm.
   (d) $E = l^2$.