II- NORMED VECTOR SPACES AND BANACH SPACES

These notes introduce the general setting of normed vector spaces and in particular Banach spaces.

1. Normed vector spaces

Metric spaces (and topological spaces) might be too general for usual applications to partial differential equations, though recent progress on this topic has shown the contrary. In particular, adding a vectorial structure on our abstract space might be convenient.

Therefore, in this chapter, E will denote a vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ (allowing for complex scalars is not only for abstract purposes; it is for example useful if we want to solve such equations as Schrodinger equation from quantum mechanics).

Definition 1.1. *A norm on E is a function denoted* $||.||$ *from E to* $\mathbb{R}^+$ *such that, for all* $x, y \in E$*, for all* $\lambda \in \mathbb{K}$

1) $||x|| = 0$ *iff* $x = 0$

2) (positive homogeneity) $||\lambda x|| = |\lambda|||x||$

3) (triangular inequality) $||x + y|| \leq ||x|| + ||y||$

A vector space E together with a norm $||.||$ is called a normed vector space. You should check immediately that if we set $d(x, y) = ||x - y||$, then d is a distance over E. This is called the distance induced by the norm $||.||$. Therefore, E with a norm is a particular case of a metric space (what about the converse?). In particular, all the notions of the previous chapter apply here.

Remark 1.2. *We have many examples of norms: (***work out these facts***)*

1) the usual euclidean norm on $\mathbb{K}^n$ *.*

2) If M is a compact metric space, let $E = C(M)$ *denotes the set of continuous functions (with values in* $\mathbb{K}$*). Define*

$$||f|| = \sup_{x \in M} |f(x)|$$

Show that this is a norm on E.

3) If (X, Σ, μ) *is a measure space, then if* $1 \leq p < \infty$, $||f||_p \equiv [\int_X |f|^p d\mu]^{\frac{1}{p}}$ *is a norm on* $E = L^p(X)$*, similarly for* $L^\infty(X)$ *and the associated norm*

$$||f||_\infty = ess \sup_{x \in X} |f(x)|$$

4) Similarly for the spaces l^∞*.*

5) If F is a linear subspace of E, then the restriction $||.||_F$ *of the norm to F is also a norm on F.*

6) If E and F are two normed vector spaces, then we can define a norm on $E \times F$ *by setting* $||(x, y||_{E \times F} = ||x||_E + ||y||_F$*.*

We have seen previously the notion of equivalent distances for metric spaces: if we have two equivalent distances on an abstract space, then the two induced topologies are the same.

For normed vector spaces, we have a similar notion of equivalent norms. Let E be a vector space. Two norms $||.||_1$ and $||.||_2$ are equivalent iff there exist $c > 0$, $C > 0$ such that for all $x \in E$, we have

$$c||x||_1 \leq ||x||_2 \leq C||x||_1.$$

It is not difficult to show that this defines an equivalence relation among norms on E. Moreover, the induced topologies are the same: in particular, the convergent or the Cauchy sequences are the same for any equivalent norm.

Though in general, in an abstract vector space E, two norms need not be equivalent, this is always the case if the dimension of E if finite. You have already seen the proof of this fact before:

Proposition 1.3. *If E is a finite dimensional vector space, then all norms are equivalent.*

Proof. **Work out this**: Fix a basis e_i, $1 \leq i \leq n$. Then any $x \in E$ has components x_i in this basis. Define $N(x) = \sqrt{\sum_i x_i^2}$. It is not difficult to see that N is a norm on E (somehow similar to the euclidean norm on $\mathbb{R}^n$).

Next let $\|.\|$ be any (other !) norm on E. Then show, using the components and Cauchy-Schwarz inequality, that there exists a constant $C > 0$ such that, for all $x \in E$, we have $\|x\| \leq CN(x)$. Then show that $\|.\|$ is continuous on E, and that it is strictly positive on the unit sphere defined from the norm N. Complete the remaining arguments. □

In particular this shows that any finite dimensional vector normed space is always complete. Moreover a finite dimensional subspace of a vector normed space is always closed. Beware that if we remove the finite dimension assumption, this is not necessarily true.

Remark 1.4. *Let F be the linear subspace of l^∞ consisting of sequences having only finitely many non zero terms. Then F is not closed.*

Indeed, let $x = (1, 1/2, 1/3, ...)$ *which is in l^∞ and not in F. Let* $x_n = (1, 1/2, 1/3, ..., 1/n, 0, , 0, ...)$ *which is in F, One can show that $x_n \to x$ in l^∞. Therefore F is not closed.*

This shows that a linear subspace F of a normed space is not necessarily closed. But this is always the case for $\bar{F}$ (its closure), i.e. $\bar{F}$ is a closed linear subspace of E (**work out this**).

Usually F needs not be a vector subspace of E. In this case, it is useful to pass to its span. We fix some vocabulary. Let $F \subset E$ be any non empty set.

We let $Vect(F)$ (the span of F) be the set of all linear combinations of elements in F, or equivalently the intersection of all linear subspaces containing F.

In the same way, we let $cl(F)$ be the intersection of all closed linear subspaces of F.

It is not difficult to show that:

$cl(F)$ is a closed linear subspace of E containing F. And that $cl(F) = \overline{Vec(F)}$. Therefore we do not loose in both notions.

One important result if given by

Theorem 1.5. *(Riesz) The unit closed ball (or the unit sphere) of a vector normed space* $(e, \|.\|)$ *is compact iff E is finite dimensional.*

Proof. Of course, if E is finite dimensional, there is nothing to prove (i.e. this is known). So let's assume that E is infinite dimensional. We shall need Riesz's Lemma (see below). We show the proof for the unit sphere $S = \{x \in E, \|x\| = 1\}$.

Let $x_1 \in S$. Since we know that E is not finite dimensional, then $Vec\{x_1\} \neq E$. Furthermore, since $Vec\{x_1\}$ is finite dimensional, it is closed. By Riesz's Lemma, there exists $x_2 \in S$, such that $\|x_1 - \lambda x_1\| \geq 3/4$ for all $\lambda \in \mathbb{K}$.

Similarly, $Vec\{x_1, x_2\} \neq E$ and again there exists $x_3 \in S$ such that

$$\|x_3 - \alpha x_1 - \beta x_2\| \geq 3/4$$

for all $\alpha, \beta \in \mathbb{K}$. Continuing, we see that there exists a sequence x_n in S such that $\|x_n - x_m\| \geq 3/4$, and thus without any convergent subsequence. Thus S cannot be compact and so also for the closed ball. □

Lemma 1.6. *(Riesz) Let E be a normed vector space, F a closed linear subspace of E, with $F \neq E$, and let α such that* $0 < \alpha < 1$. *Then there exists $x_\alpha \in E$, such that* $\|x_\alpha\| = 1$ *and* $\|x_\alpha - y\| > \alpha$ *for all $y \in F$.*

Proof. Since $F \neq E$, there exists $x \in E - F$. Since F is closed, $d = d(x, F) > 0$. Thus also $d < d\alpha^{-1}$. Therefore there exists $z \in F$ such that $||x - z|| < d\alpha^{-1}$.

We set $x_\alpha = \frac{x-z}{||x-z||}$. For course this is a unit vector, and for all $y \in F$, we have

$$||x_\alpha - y|| = ||\frac{x - z}{||x - z||} - y|| = \frac{1}{||x - z||}||x - (z + ||x - z||y)|| > (\alpha d^{-1})d = \alpha$$

using the fact that F is a linear subspace.

□

From Basic courses, you know that if E is a finite dimensional normed vector space, then any linear form on E is continuous. This is not necessarily true if E is infinite dimensional, see exercices.

However, a good point is that linear continuous applications are equivalent to a boundedness principle in the following sense:

Theorem 1.7. *Let $(E, ||.||_E)$ and $(F, ||.||_F)$ be two normed vector spaces. Let $f : E \to F$ be a linear application. Then 1) is equivalent to 2) and equivalent to 3)*

1) f is continuous at 0

2) f is uniformly continuous

3) f is bounded: there exists $M > 0$ such that for all $x \in E$, we have $||f(x)||_F \leq M||x||_E$.

Proof. **Work out this**

□

This is an important result. Let us introduce $L_c(E; F)$ for the set of all continuous and linear applications from E to F (withe fixed norms on each...). Then we can define a norm on this space by setting

$$|||f||| = \sup_{x \in E, ||x||=1} ||f(x)|| = \sup_{x \in E, ||x|| \leq 1} ||f(x)|| = \sup_{x \in E, x \neq 0} \frac{||f(x)||}{||x||}$$

Check out this point

Also note that the sup is not necessarily a max.

As particular examples of linear applications, we will now consider the case of linear forms (i.e. valued in $\mathbb{K}$), together with the notion of hyperplanes (we shall say more on this later on).

First recall that if E is a vector space, and if V and W are two vector subspaces of E, then we say that they are supplementary if we can write $E = V + W$. We say that they form a direct sum (of E) if they are supplementary and $V \cap W = \{0\}$. Then we write $E = V \oplus W$.

It is not difficult to show that if this is the case, then for any $x \in E$, there exists unique v and w in V and W resp. such that $x = v + w$. We denote $v = p_V(x)$ and $w = p_W(x)$. It is not difficult to show also that this defines two linear applications p_V and p_W which are the corresponding projections.

Furthermore, if W' is another vector subspace such that $E = V \oplus W'$ then W and W' are isomorphic (use the projections).

Therefore all subspaces which are in direct sum with a fixed subspace V are all isomorphic between them, and thus have the same dimension. This number is by definition the co-dimension of V.

By definition an hyperplane is a vector subspace of co-dimension 1.

One can show :

Proposition 1.8. *The hyperplanes H of a vector space E are the kernels of the non null linear forms on E. Furthermore, if E is a normed vector space, and $H = Ker\ (l)$ for some non null linear form, then H is closed iff l is continuous.*

Proof. **Work out this** □

Since continuity plays a key role, this is why we introduce, if E is a normed vector space, the (topological) dual of E, denoted by E' which is nothing else than $L_c(E;\mathbb{K})$.

Later on , we shall prove the following result (a particular case of Hahn-Banach theorem)

Theorem 1.9. *Let E be a normed vector space and V a vector subspace of E. If $m \in V'$ (for the induced norm), then there exists an extension $l \in E'$ of m not necessarily unique and with the same norm*

$$\forall v \in V, m(v) = l(v) \text{ and } \|l\|_{E'} = \|m\|_{V'}$$

This Theorem implies the following

Proposition 1.10. *Let E be a normed vector space, and $x \in E$. If for all $l \in E'$, we have $l(x) = 0$, then $x = 0$.*

2. Banach Spaces

Definition 2.1. *A Banach space is a normed vector space and complete w.r.t. the metric associated with the norm.*

Examples:
- Any finite dimensional normed vector space is a Banach space.
- if X is a compact metric space, then $C(X)$ is a Banach space.
- if (X, Σ, μ) is a measure space, then $L^p(X)$ is a Banach space.
- l^p is a Banach space.
- if E is a Banach space, and if F is a linear subspace of E, then F is Banach iff F is closed.

A classical application is this:

Proposition 2.2. *If E is a vector normed space, and if F is a Banach space, then $L_c(E;F)$ is a Banach space.*

Proposition 2.3. *If E is a Banach space, and if x_n is a sequence of elements of E, then if the series $\sum_n \|x_n\|$ converges, then the series $\sum_n x_n$ converges also.*

3. Differential Equations in Banach spaces

We start with some facts about integrals of functions of a real variable, but valued in a normed vector space.

Proposition 3.1. *Let $a < b$ two real numbers. Let B be a Banach space. Let $f \in C([a,b];B)$. Let $h = (t_1, ..., t_N)$ be a subdivision of $[a,b]$. Set $I_h(f) = \sum_{n=1}^{N-1}(t_{n+1} - t_n)f(t_n)$. Then there exists a vector $I(f) \in B$ such that $I_h(f) \to I(f)$ when $|h| \to 0$ (here $|h$ is the maximal length of the subdivision), which is by definition the integral of f over $[a,b]$: $\int_a^b f(t)dt$. Set also $\int_b^a = -\int_a^b$. Then one has*

$$\|\int_a^b f\| \leq |b-a| \sup |f|$$

and the usual Chasles relation. I is continuous from C^0 into B.

Let us recall the notion of the derivative of a function from an interval J into a normed vector space. In particular if the derivative is zero, then this function is a constant.

One can show, that if J is an interval and B a Banach space, if $f \in C(J;B)$, then all primitives (antiderivatives) of f are given by functions $F \in C^1(J;B)$ such that $F(t) = x + \int_a^t f(s)ds$, for any x and a.

By using this and a fixed point theorem, one can show

Theorem 3.2. *(Cauchy-Lipschitz) Let B be a Banach space over $\mathbb{K}$ and J an interval. Let $F : J \times B \to B$ be continuous and globally Lipschitz w.r.t. the second variable:*

$$\text{there exists } L > 0, \forall t \in J, \forall x, y \in B, \|F(t,x) - F(t,y)\| \leq L\|x - y\|$$

Then: for any $t_0 \in J$, for any $x_0 \in B$, there exists a unique solution $\phi \in C^1(J; B)$ of the ode $y' = F(t, y)$ such that $\phi(t_0) = x_0$.