I- SOME KNOWN FACTS

This Chapter describes very quickly the basics facts on metric spaces. **You can work out the proofs of the results below**, which are all standard from a classical course on topology. We will shift from Chapter 2 to a special case of metric space, i.e. normed vector spaces. We also recall some basic elements from Lebesgue integration. For the measure theory, it is **recommended** to read a classical course on this topic. Finally, we have given some results related to sequences, which might be useful. Some of the notions below were taken from textbooks or lectures notes by Poupaud, Hirsch, Lacombe, Rudin.

1. Metric spaces

Definition 1.1. *A metric space (E, d) is a set E together with an application $d : E \times E \to \mathbb{R}^+$ called a distance or a metric, such that, for all $x, y, z \in E$:*
1) $d(x, y) = 0$ iff $x = y$.
2) $d(x, y) = d(y, x)$
3) $d(x, z) \leq d(x, y) + d(y, z)$

Note that E is not assumed to be a vector space. Then we have the following sets:

$$B_c(x; r) \text{ and } B(x; r)$$

which are called open balls and closed balls.

A metric space (E, d) becomes a topological space if we define a suitable notion of open sets.

An open set O in a metric space (E, d) is a subset of E such that, for all $x \in O$, there exists an open ball $B(x; r) \subset O$.

One can show that if $\mathcal{O}$ denotes the collection of such open sets, then $(E, \mathcal{O})$ is a topological set: this is the topology induced by the metric d. Of course, if we change the metric d, then we get another collection of open sets. But if the metrics are equivalent, then we get exactly the same topology. Two distances d and d' on a set E are said equivalent if there exist two non negative constants c, c', such that for all $x, y \in E$, $cd(x, y) \leq d'(x, y) \leq c'd(x, y)$.

In any case, you must remember that any (finite or not) union of open sets is an open set, and any finite intersection of open sets is again an open set.

A neighborhood of $x \in E$ is a set containing an open set containing x.

Once we have defined open sets (in a metric space), then we can define a closed set: a closed subset $F \subset E$ is said to be closed if its complementary $E - F$ is open. Therefore, one can show that any intersection of closed sets is a closed set, while any finite union of closed sets is again a closed set.

Definition 1.2. *Let $X \subset (E, d)$. Then the interior X° is the biggest open set included in X. Its closure (adherence) $\bar{X}$ is the smallest closed set containing X. Its boundary is $\partial X = \bar{X} - X^\circ$.*
A set X is said to be dense in E if $\bar{X} = E$.

The closure of X is in fact the set of all its closure points, in the following sense: a point $x \in E$ is said be a closure point of X if for any $\varepsilon > 0$, there exists a $y \in X$, such that $d(x, y) < \varepsilon$. (Equivalently, see below, if there exists a sequence $y_n \in X$ such that $y_n \to x$).

The notion of distance or metric is between points of E. We can also define a metric between subsets. For example, if $x \in E$, and if $A \subset E$, then we set

$$d(x, A) = \inf_{y \in A} d(x, y)$$

and similarly if $\subset E$ and $B \subset E$, we set

$$d(A, B) = \inf_{x \in A, y \in B} d(x, y).$$

There many ways to construct new distances on new sets, starting from old ones. One typical example is provided by the so called product metric: if (E, d) and (F, d') are two metric spaces, then we can define a new metric $\tilde{d}$ on $E \times F$ by the following formula

$$\tilde{d}((x, x'), (y, y')) = d(x, y) + d'(x', y')$$

For checking some topological properties, sequences are useful. Let us first of all recall

Definition 1.3. *Let a_n be a sequence in a metric space (E, d), and let $a \in E$. We say that a_n converges to a, $a_n \to a$ or $\lim_n a_n = a$ if $d(a_n, a) \to 0$.*

It is of course understood that n goes to infinity. In fact, the above notion is topological in the sense that one can show that a_n converges to a iff for any neighborhood V of a, there exists an integer N such that, for any $n \geq N$, one has $a_n \in V$.

In case a sequence does not converge, it might be useful to pass to a subsequence:

Definition 1.4. *A point $x \in E$ is said to be a limit point or an adherent point of a sequence a_n if there exists a subsequence such that $a_{n_k} \to x$.*

One can show for example that a set F is closed if its contains the limits of any convergent sequence of F.

We can now go to the continuity of an application f from a metric space (E, d) to another one (E', d'). Since these two spaces are particular topological spaces, recall that for an $x \in E$, f is said to be continuous at x iff for any neighborhood W of $f(x)$, there exists a neighborhood $V \subset E$ of x such that $f(V) \subset W$, that is the inverse image of any neighborhood of $f(x)$ is a neighborhood of x.

On can show

Proposition 1.5. *f is continuous at x iff we have 1) iff we have 2).*

1) For any sequence $a_n \to x$, one has $f(a_n) \to f(x)$.

2) For all $\varepsilon > 0$, there exists $\delta > 0$, such that for all y, $d(x, y) < \delta$, then $d'(f(x), f(y)) < \varepsilon$.

A function f which is continuous at any point x is said to be continuous. We let $C(E; E')$ be the set of all continuous functions from E to E'. It is not difficult to show that an an application f is continuous iff the inverse image of any open set is an open set. This is also equivalent to say that the inverse image of any closed set is a closed set.

Moreover, continuity is preserved by composition.

We can now go to sequences of functions: let $f_n : E \to E'$ and $f : E \to E'$. Recall that $f_n \to f$ (simply or pointwise) iff for any fixed $x \in E$, one has $f_n(x) \to f(x)$. This is called simple or pointwise convergence, For many purposes, this is not a convenient notion. For example, properties of functions f_n are not conserved by this type of convergence. One useful notion which is better is the uniform convergence:

Definition 1.6. *We say that f_n converges uniformly to f, $f_n \to_{u.c} f$ iff*

$$\sup_{x \in E} d'(f_n(x), f(x)) \to 0.$$

From previous courses, you know that the uniform limit of continuous functions is a continuous function (which is not true if we assume only a pointwise convergence).

Finally, another useful notion is the uniform continuity:

Definition 1.7. *A function $f : E \to E'$ is said to be uniformly continuous if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in E$, then $d'(f(x), f(y)) > \varepsilon$.*

Note the position of variable x.

For many reasons, basic metric spaces are not enough for applications. For example, suppose that you have a numerical algorithm which generates a sequence a_n and you wish to show that it converges (to some a). Of course, usually you do not know a. Therefore, we need a criterion which will entail the convergence of our sequence. The best one is given by

Definition 1.8. *A sequence a_n is said to be a Cauchy sequence iff for all $\varepsilon > 0$, there exists an integer N such that for all $n \geq N$, $m \geq N$, then $d(a_n, a_m) < \varepsilon$.*

The point is to notice that first of all, any convergent sequence is automatically a Cauchy sequence. Therefore, it makes sense to test the Cauchy condition above first. However, even if a sequence is a Cauchy sequence, it does not mean that it will converge. This is why we need

Definition 1.9. *A metric space (E, d) is said to be complete if any Cauchy sequence is convergent (in E).*

You have already seen in previous classes many examples of complete or non complete spaces. Let us add that if (E, d) is a complete metric ace, and if F is a subset of E, then F is complete iff F is closed. The notion of completeness is useful for many applications. We provide two examples

Theorem 1.10. *(Extension of a uniformly continuous map) Let (E, d) and (E', d') be two metric spaces, with E' complete. Let D be a dense set of E and let $f : D \to E'$ be uniformly continuous on D. Then there exists an unique uniformly continuous extension $\tilde{f}$ of f on the whole of E.*

Theorem 1.11. *(Picard fixed point theorem) Let (E, d) be a complete metric space. Let f be a strictly contractive map of E, that is there exists $0 < k < 1$ such that for all $x, y \in E$, $d(f(x), f(y)) \leq kd(x, y)$. Then there exists a unique fixed point of f in E, that is a point $a \in E$ such that $f(a) = a$. This fixed point can be obtained as the limit of the sequence a_n, with $a_{n+1} = f(a_n)$, for any choice of initial value a_0. We have $d(a_n, a) \leq k^n d(a_0, a)$.*

Another way to get convergent sequences is based on compactness arguments. Recall for example Bolzano Weierstrass theorem. Compactness is a topological property.

Definition 1.12. *Let E be any topological space. Then E is said compact if for any covering of E by open sets, we can extract a finite covering. We say that E is sequentially compact if from any sequence of E, we can extract a convergent sequence (in E). We say that it is separable if there exists a countable dense subset. A subset X is said to be relatively compact if its closure $\bar{X}$ is compact for the induced topology.*

Finally, if E is a metric space, E is said to be pre-compact if for any $\varepsilon > 0$, there exists a finite number of points in E, say x_1, ..., x_n such that E is exactly the union of the open balls $B(x_i; \varepsilon)$.

Note that any subset of a separable space is also separable. It is not difficult also to show that a compact metric space is separable.

These are general definitions. However, one can show (work out the proof which is not so easy) the following result

Proposition 1.13. *Let (E, d) be a metric space. Then 1) is equivalent to 2) and equivalent to 3) .*

1) (E, d) is compact

2) (E, d) is sequentially compact

3) (E, d) is pre-compact and complete

On can also show that

Proposition 1.14. *Let X be a subset of a complete metric space (E, d). Assume that X is pre-compact. Then it is relatively compact.*

Proposition 1.15. *Let (E, d) and $(E', d'$ two metric spaces, and $f \in C(E; E')$. Then if E is compact, $f(E)$ is also compact.*

You have certainly seen previously Tykhonov theorem: it says that any product of compact topological spaces is also compact (see any book on topology). In the special case of two compact metric spaces E and E', then the product $E \times E'$ is also compact for the product metric.

Finally, other useful properties are:

- A real function on a compact metric space is bounded and attains its bounds.

- A compact metric space is automatically bounded. Here we need to recall that by definition a subset A of a metric space E is bounded if there exists $c > 0$ such that $d(x, y) \leq c$ for all $x, y \in A$.

- A continuous function from a compact metric space is automatically uniformly continous.

Later on, we shall see and prove the following important result

Theorem 1.16. *(Baire) If (E, d) is a complete metric space, and if $E = \cup_{i=1}^{\infty} F_j$, where the F_j are closed, then at least one of the F_j contains an open ball.*

2. Lebesgue integration

Consider the space $C([a, b]$ of all (real or continuous) functions, together with the uniform metric d_∞. One can show that this space is complete (why?).

Now consider the following metrics on $C([a, b])$, for $1 \leq p < \infty$:

$$d_p = [\int_a^b |f(x) - g(x)|dx]^{\frac{1}{p}},$$

This is also a metric (why), but $C([a, b])$ with this metric is not complete. This is one reason why Riemann integrals are not suitable and that we need to shift to Lebesgue integration.

We consider a general setting of an abstract space X (instead of the interval $[a, b]$). We will also consider extended real numbers with the convention that $0.\infty = 0.(-\infty) = 0$.

Recall that a $\sigma-$algebra or a $\sigma-$field on X is a collection Σ of subsets of X, containing the full set X and the empty set , which is stable taking complements and by countable union.

An element of Σ is called a measurable set.

Next assume that we have such a couple (X, Σ) of an abstract set X together with a $\sigma-$algebra. Then a measure μ is a map from Σ to $\bar{\mathbb{R}}^+$ such that the image of the empty set is 0 and countably additive (i.e. the image of a countable union of disjoints elements of Σ is the sum of the images).

Then (X, Σ, μ) is called a measure space.

A set $N \in \Sigma$ such $\mu(N) = 0$ is said to have measure zero or is a null set. We have the notion of "almost everywhere" ...

Two examples are the counting measure and the Lebesgue measure.

• Counting measure: take $X = \mathbb{N}$, and let Σ_c be all the subsets of $\mathbb{N}$. Then for any $S \subset \mathbb{N}$, let $\mu_c(S) = \ card\, S$.

• Lebesgue measure: there is a $\sigma-$algebra Σ_L on $\mathbb{R}$, containing any finite interval $[a, b]$, and a measure μ_L on Σ_L such that $\mu_L([a, b]) = b - a = l([a, b])$. The sets of measure zero of this space are exactly the sets A such that: for any $\varepsilon > 0$, there exists a countable collection of intervals $I_j \subset \mathbb{R}$ such that $A \subset \cup_{i=1}^{\infty} I_j$ and $\sum_j l(I_j) < \varepsilon$. This is the Lebesgue measure, and the sets in Σ_L are said to be Lebesgue measurable.

Now assume that we have a measure space (X, Σ, μ). We describe how to construct an integral of a function $f : X \to \bar{\mathbb{R}}$. As usual we start with so called simple functions. First, for any subset $A \subset X$, we let χ_A denote its characteristic function. A function $f : X \to \mathbb{R}$ is said to be simple if it can be written as $f = \sum_{j=1}^{k} \alpha_j \chi_{S_j}$, for some $k \in \mathbb{N}$ and some $S_j \in \Sigma$, $j = 1, ..., k$.

If f is non negative and simple, then by definition its integral (w.r.t. μ and over X) is defined by

$$\int_X d\mu = \sum_{j=1}^{k} \alpha_j \mu(S_j).$$

Eventually $\mu(S_j) = \infty$, and eventually $\int_X f d\mu = \infty$ also.

To define the integral for general functions, we need to restrict to so called measurable functions. A function $f : X \to \bar{\mathbb{R}}$ is said to be measurable if for every $\alpha \in \mathbb{R}$, the set $\{x \in X, f(x) > \alpha\}$ belongs to Σ.

This notion of measurability is stable by passing to absolute value, and to minus and plus operations: i.e. $|f|$, $f^{\mp}$ are also measurable.

Assume now that f is measurable and non negative. Then we define

$$\int_X f d\mu = \sup\{\int_X f d\mu,\ g \text{ simple and } 0 \leq g \leq f\}$$

If f is measurable and $\int_X |f| d\mu < \infty$, then we say that f is integrable and we define

$$\int_X f d\mu = \int_X f^+ - \int_X f^- d\mu.$$

The extension to complex valued functions is similar. Finally we can define also the integral over an element of Σ (by extending by 0 outside).

We let $\mathcal{L}^1(X)$ be the set of all integrable functions. In particular for the counting measure example, we recover the usual $l^1(\mathbb{N})$ space, while for the Lebesgue case, we recover the space $\mathcal{L}^1(\mathbb{R}^k)$ of Lebesgue integrable functions. Let us note immediately that on compact intervals of $\mathbb{R}$, a function which is bounded and Riemann integrable is also Lebesgue integrable, with the same integrals.

In the general case, the notion of integral shares usual properties such as: linearity, monotonicity .. Furthermore if $f = 0$ a.e., then its integral is zero. This is useful in particular if we allow bounds not eveywhere but only a.e.

If f is measurable and there exists a constant c such that $f(x) \leq c$, then we can define its essential supremum as $ess\ \sup f = \inf\{M, f(x) \leq M a.e.\}$. It follows that $f(x) \leq\ ess\ \sup f$ a.e. Similarly, we can define the essential infimum.

Then a function f is essentially bounded if there exists a constant c such that $f(x) \leq c$ a.e.

Next introduce the following definition

$$d_1(f, g) = \int_X |f - g| d\mu.$$

for all functions f, g in $\mathcal{L}^1(X)$. This is not quite a metric. So we pass to quotient, using the relation $\equiv$ to identify functions which agree a.e. The quotient space is defined as $L^1(X)$, and we can define the integral a (class of) function.

More generally, letting

$$\mathcal{L}^p(X) = \{f, f \text{ measurable and } [\int_X |f| d\mu]^{\frac{1}{p}} < \infty\}$$

for $1 \leq p < \infty$, and

$$\mathcal{L}^\infty(X) = \{f, f \text{ measurable and } ess\ \sup |f| < \infty\}$$

we can introduce the spaces $L^p(X)$ and $L^\infty(X)$.

Recall that for any measurable functions f and g, then (eventually with infinite values):

- Minkowski's inequality: for $1 \leq p < \infty$

$$[\int_X |f+g|^p d\mu]^{\frac{1}{p}} \leq [\int_X |f|^p d\mu]^{\frac{1}{p}} + [\int_X |g|^p]^{\frac{1}{q}}$$
$$ess\ \sup|f+g| \leq ess\ \sup|f| + \ ess\ \sup|g|$$

- Holder's inequality for $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$

$$\int +X|fg|d\mu \leq [\int_X |f|^p d\mu]^{\frac{1}{p}} [\int_X |g|^q]^{\frac{1}{q}}$$

$$\int +X|fg|d\mu \leq \ ess\ \sup|f| \int_X |g|d\mu$$

It follows that $L^p(X)$ is a metric space, and that

$$d_p(f,g) = [\int_X |f-g|^p]^{\frac{1}{p}}, 1 \leq p < \infty,\ ess\ \sup|f-g|\ if\ p = \infty$$

is a metric on $L^p(X)$.

It can be shown that they are also complete metric spaces. Finally note that we recover usual inequalities in the case of the counting measure.

3. More on sequences

Let X be any set. X is said to be countably infinite if there exists a bijection ϕ from $\mathbb{N}$ to X, i.e. we can order X as $X = \{\phi(0), \phi(1), ..\}$, with $\phi(n) \neq \phi(p)$ if $n \neq p$. We can also use the notation $\phi(n) = x_n$. A set is countable if it either finite or countably infinite.

For example, $\mathbb{N}$, $\mathbb{N}^2$ are countable.

It can be shown that a nonempty set X is countable iff there exists a surjection from $\mathbb{N}$ to X, which is equivalent to the existence of an injection from X to $\mathbb{N}$. A finite product or a countable union of countable sets is again countable.

We now describe one of the most important tools in analysis, i.e. the diagonal procedure or extraction.

First of all, if x_n is a given sequence, recall that a subsequence is a sequence of the form x_{n_k}, where n_k is a strictly increasing sequence of integers. It can be also written as $x_{\phi(k)}$ where $\phi : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function. If we let $\phi(\mathbb{N}) = A$, we can denote this subsequence by $(x_n)_{n\in A}$.

Proposition 3.1. *Let (X_p, d_p) be a sequence of metric spaces, and let for every $p \in \mathbb{N}$, $(x_{n,p})_{n\in\mathbb{N}}$ be a sequence in X_p. Assume that for every $p \in \mathbb{N}$, the set $\{x_{n,p}, n \in \mathbb{N}\}$ is relatively compact in X_p.*

Then there exists a strictly increasing function $\phi : \mathbb{N} \to \mathbb{N}$ such that for every $p \in \mathbb{N}$, the sequence $(x_{\phi(n),p})_{n\in\mathbb{N}}$ converges in X_p.

Let us recall that a subset Y of a metric space X is relatively compact if there exists a compact set K of X such that $Y \subset K$ or equivalently if the closure of Y in X is compact. This is also equivalent to: Y is relatively compact iff every sequence in Y has a subsequence that converges in X (the limit not being necessarily in Y).

The important point is that the function ϕ does not depend on p.

Proof: By induction, due to the relative compactness, we can construct a decreasing subsequence A_n of infinite subsets of $\mathbb{N}$ such that, for every $p \in \mathbb{N}$, the sequence $(X_{n,p})_{n\in A_p}$ converges in X_p. Here we use the **diagonal procedure**: it consists in defining the map ϕ by setting

$$\phi(p) = \text{ the } (p+1) - st \text{ element of } A_p.$$

Thus $\phi(p+1)$ is strictly greater than the $(p+1)$-st element of A_{p+1}, which in turn is greater than the $(p+1)$-st element of $A - p$, which is $\phi(p)$. Thus ϕ is strictly increasing. Moreover, for every $p \in \mathbb{N}$, the sequence $(x_{\phi(n),p})_{n\geq p}$ is a subsequence of the sequence $(x_{n,p})_{n\in A_p}$ because if $n \geq p$, we have $\phi(n) \in A_n \subset A_p$. Thus the sequence $(x_{\phi(n),p})_{n\in\mathbb{N}}$ converges.

It is then possible to give a proof of Tychonoff's theorem. For this purpose, let (X_p, d_p) be a sequence of metric spaces. Put $X = \prod_{p\in\mathbb{N}} X_p$, which is the set of sequences $x = (x_p)_{p\in\mathbb{N}}$ such that $x_p \in X_p$ for all $p \in \mathbb{N}$. The followinf

$$d(x, y) = \sum_{p=0}^{\infty} 2^{-p} \min(d_p(x_p, y_p), 1)$$

defines a metric d on X. This is the product distance on X. For this metric, a sequence x^n of points in X converges to a point $x \in X$ iff $lim_n x_p^n = x_p$ for any $p \in \mathbb{N}$. If the metric spaces (X_p, d_p) are all the same (Y, δ), we write $X = Y^{\mathbb{N}}$. Then X is the set of sequences in X or the set of maps from $\mathbb{N}$ into Y, with the pointwise convergence. Then we get the Tychonoff's theorem: if X_p is a sequence of compact metric spaces, then the product space is also compact.