Functional Analysis

Exercices I- Metric spaces

1. Let $0 \leq \varrho < r$ and $x, y \in E$. Show that $B_c(x;\varrho) \subset B(x;r)$. Show also that if $d(x,y) < r - \varrho$ then $B_c(y;\varrho) \subset B(x,r)$.
2. Show that open balls are open sets and that closed balls are closed sets (for the topology induced by the metric d on E).
3. Show that a set is open iff it is a neighborhood of all its elements.
4. Metric spaces setting is a very large one, and can be misleading. For example, the interior of a closed ball is not necessarily the open ball with the same center and same radius. Let (E,d) be a metric space, $x \in E$ and $r > 0$. Find the relationship (inclusions or equalities) between the following sets
$$(B(x;r))^\circ,\ B(x;r), \overline{B(x;r)},\ (B_c(x;r))^\circ,\ B(x;r), \overline{B_c(x;r)}$$
Let $E = \{-2\} \cup [-1,2]$ with the distance $d(x,y) = |x-y|$. Show that
$$B_c(1;1) = [0,2],\ B(1;1) =]0;2[, (B_c(1;1))^\circ =]0,2[$$
Find also $B_c(-1;1)$, $B(-1;1)$ and $\overline{B(-1;1)}$. Comments ?
5. Show that x is not an adherent (or limit point) point of a sequence $(a_n)_n$ iff there exists an $\varepsilon > 0$ such that $B(x;\varepsilon)$ does contain only a finite number of elements of the sequence (a_n).
6. Let X be a dense set of a metric space (E,d). Show that, for all $x \in E$, there exists a sequence x_n of X such that $x_n \to x$.
7. Let F be a closed set of a metric space (E,d), and $x \in E$. Show that $x \in F$ iff $d(x,F) = 0$.
8. Show that the (local) continuity is preserved by composition between continuous functions in metric spaces.
9. Let $E,d)$ be a metric space. Show that $\tilde{d} = d_{E\times E}$ is a metric on $E \times E$. We equip $E \times E$ with this metric, and $\mathbb{R}^+$ with the usual distance. Show that $\tilde{d}$ is a uniformly continuous function.
10. Let $E = \mathbb{R}$ equipped with the usual distance. Show that $f(x) = x^2$ is not uniformly continuous. Show that $d'(x,y) = |\arctan(x) - \arctan(y)|$ is also a distance on $\mathbb{R}$. Is f uniformly continuous w.r.t. this distance ?
11. Show that a convergent sequence is a Cauchy sequence. Assume now that an extracted sequence of a Cauhcy sequence converges. Then show that the whole sequence converges.
Let now $E = [0,\infty[$ and $d(x,y) = |\arctan(x) - \arctan(y)|$. Show that this a metric space. Show that $a_n = n^2$ is a Cauchy sequence. Does it converge ?
12. Let $E = \{f : \mathbb{R} \to \mathbb{R}, f \geq 0, f \in C^0(\mathbb{R};\mathbb{R}), f \text{ bounded}\}$, together with the distance $d(f,g) = \sup_{x\in\mathbb{R}} |f((x) - g(x)|$.
Let $g \in E$ and $k \in C^0(\mathbb{R}^2;\mathbb{R}^+)$ such that there exists $\phi : \mathbb{R} \to \mathbb{R}$ integrable with
$$\forall x, y \in \mathbb{R}, k(x,y) \leq \phi(y),\ \int_{\mathbb{R}} \phi(t)dt < 1$$

We look for $f \in E$ such that

$$\forall x \in \mathbb{R}, f(x) - \int_{\mathbb{R}} k(x,y)f(y)dy = g(x)$$

(a) Show that (E,d) is a complete metric space.

(b) Show that for all $u \in E$, $K(u) : x \in \mathbb{R} \mapsto \int_{\mathbb{R}} k(x,y)u(y)dy \in E$.

(c) Using a fixed point, show that we have a unique solution f to the problem.

13. Let X be a subset of a metric space (E,d). Show that if X is pre-compact, then this is also the case for its closure $\bar{X}$.