首届致远学术节
学生科研成果展示

Optimal Overbooking Limits in Two-stage Capacity Allocation with Random Reservations and Show-ups

Keywords: overbooking, non-convex, stochastic gradient; **Independent Research**, supervised by Professor **Xin Chen** at UIUC

1. Background

- ✓ Overbooking is an effective policy in hotel booking, airline, retailer and so on. It can increases the revenue.
- ✓ It is solved by optimization.
- ✓ It has lots of variations, e.g., **static** vs. dynamic; deterministic vs. **stochastic**; **multi classes** vs. single class. All these factors bring the **difficulty** in solving the model

2. Problem Statement

Decision Variables ——**Overbooking Limits**

First period: reservation period

✓positive dependent/independent random reservations (discrete)

✓random show ups and cancellations at the end of this period

Second period: service period

✓Multiple reservation and inventory classes

✓Allocation: network revenue management

3. Nonconvex Optimization Model

$$\max_{u \geq x} G(\boldsymbol{u}) = \mathbb{E}_{\Xi}\left[\boldsymbol{r}^T(\boldsymbol{u} \wedge (\boldsymbol{x}+\Xi)) - \boldsymbol{r}^T\boldsymbol{x}\right] - \mathbb{E}_{\Xi,Z}\left[\boldsymbol{q}^T\left((\boldsymbol{u} \wedge (\boldsymbol{x}+\Xi)) - \boldsymbol{Z}\right)\right] + \mathbb{E}_{\Xi,Z}[V_0(\boldsymbol{Z})]$$

where

$$(z_1, \dots, z_n) = \left(Z_1(u_1 \wedge (x_1 + \xi_1)), \dots, Z_n(u_n \wedge (x_n + \xi_n))\right)$$

$$V_0(\boldsymbol{z}) = \max \sum_{i=1}^{n} \sum_{j=0}^{m} a_{ij} y_{ij}$$

$$\text{s.t.} \quad \sum_{j=0}^{m} y_{ij} = z_i, i = 1, \dots, n,$$

$$\sum_{i=1}^{n} y_{ij} \leq c_j, j = 0, \dots, m,$$

$$y_{ij} \geq 0, i = 1, \dots, n, \ j = 0, \dots, m.$$

4. Transformed Equivalent Convex Model[2]

$$f(\boldsymbol{u} \wedge (\boldsymbol{x}+\Xi)) = \boldsymbol{r}^T(\boldsymbol{u} \wedge (\boldsymbol{x}+\Xi)) - \boldsymbol{r}^T\boldsymbol{x} - \boldsymbol{q}^T(\boldsymbol{u} \wedge (\boldsymbol{x}+\Xi)) + (\boldsymbol{q} \times \boldsymbol{p})^T(\boldsymbol{u} \wedge (\boldsymbol{x}+\Xi)) + g(\boldsymbol{u} \wedge (\boldsymbol{x}+\Xi))$$

$$\max \mathbb{E}_{\Xi}\left[f\left(v_1(\Xi_1), \dots, v_n(\Xi_n)\right)\right]$$

$$s.t. \quad v_i(\xi_i) \leq x_i + \xi_i \quad \forall \xi_i \in \mathcal{X}_i, \ \forall i = 1, \dots n$$

$$(\boldsymbol{x}, v_1(\xi_1), \dots, v_n(\xi_n)) \in \mathcal{A}^{\Xi} \quad \forall \xi \in \mathcal{X}$$

where $\mathcal{A}^{\Xi} = \{(\boldsymbol{x}, \boldsymbol{u} \wedge (\boldsymbol{x}+\boldsymbol{\xi})) | \boldsymbol{u} \geq \boldsymbol{x}, \boldsymbol{\xi} \in \mathcal{X}\}$. And we require $v_i(\cdot)$ is measurable.

5. Structural Properties

*LEMMA 1. Function $V_0(\boldsymbol{Z})$ is **submodular** with respect to $(Z_1, \dots, Z_n)$.*

*LEMMA 2. $V_0(\boldsymbol{Z})$ **is jointly concave** in $Z_1, \dots, Z_n$ and $c_1, \dots, c_m$.*

*LEMMA 3. $g(\boldsymbol{u}) = \mathbb{E}_Z[V_0(\boldsymbol{Z}(\boldsymbol{u}))]$ is **twice continuously differentiable** with regard to $\boldsymbol{u}$.*

*THEOREM. For each $i = 1, \dots n$, the nonnegative random variable $\{Z_i(u_i) | u_i \geq 0\}$ has the semigroup property with respect to the parameter u_i, then the function, $g(\boldsymbol{u})$, **is component-wise concave** in each u_i, $i = 1, \dots n$, and **submodular** in $(u_1, \dots, u_n)$.*

6. Numerical Experiments

Experiment	Problem	Algorithm	Parameter: Random Reservations	Parameter: Capacity	Optimal booking limits			
1	original	Stochastic Gradient	None	unknow	120	118	117	15
2	original	Stochastic Gradient	None	100	114	112	111	109
3	original	Stochastic Gradient	None	105	120	118	117	115
4	original	Stochastic Gradient	None	110	125	123	122	119
5	random reservations	Heuristic	[120, 130]	100	114	112	111	109
6	random reservations	Heuristic	{120, 121...,130}	110	123	122	121	119
7	random reservations	Heuristic	{120, 121...,124}	110	122	121	121	120
8	random reservations	Transformed	{120, 121...,130}	110	125	123	122	120
9	random reservations	Transformed	{120, 121...,124}	110	124	123	121	120

The **stochastic gradient algorithm** is coded in **Python using Gurobi** to solve the LP problems, implemented on **Intel Core i5 CPU 2.4GHZ and 8GB memory**. Adaptive Step Size was not considered here, we used $bu = 1/k$ for simplicity. Each iteration solves $n+1$ network problems (linear program with $O(mn+n)$ decision variables and $O(m+n+1)$ constraints.

The algorithm requires a sequence of step sizes, $\{b_k\}$, satisfying $\sum_{k=0}^{\infty} b_k = +\infty$, $\sum_{k=0}^{\infty} b_k^2 < +\infty$; Then, the algorithm proceeds as follows:

Step 0. Initialize: $k = 1$ and $\boldsymbol{u}^k := \boldsymbol{x}$.

Step 1. Get the stochastic gradient:

- Randomly generate a new vector Ξ^k.
- Randomly generate a new vector $\boldsymbol{Z}\left(\boldsymbol{u}^k \wedge (\boldsymbol{x} + \Xi^k)\right)$.
- Compute the gradient estimate $\boldsymbol{D}^k$.

Step 2. Compute the $\boldsymbol{u}^{k+1} = \prod(\boldsymbol{u}^k + b_k\boldsymbol{D}^k)$, where $\prod(\cdot)$ is the projection of $\boldsymbol{u}^k + b_k\boldsymbol{D}^k$ onto $\{\boldsymbol{u}: \boldsymbol{u} \geq \boldsymbol{x}\}$.

Step 3. Set $k := k+1$ and GOTO Step 1.

$$D_i^k = \begin{cases} r_i - q_i(1-p_i) + p_i\left(V_0(\boldsymbol{z}_i^k + \boldsymbol{e}_i) - V_0(\boldsymbol{z}_i^k)\right), & \xi_i > u_i - x_i \\ \min\{0, r_i - q_i(1-p_i) + p_i\left(V_0(\boldsymbol{z}_i^k + \boldsymbol{e}_i) - V_0(\boldsymbol{z}_i^k)\right)\}, & \xi_i \leq u_i - x_i \end{cases}$$

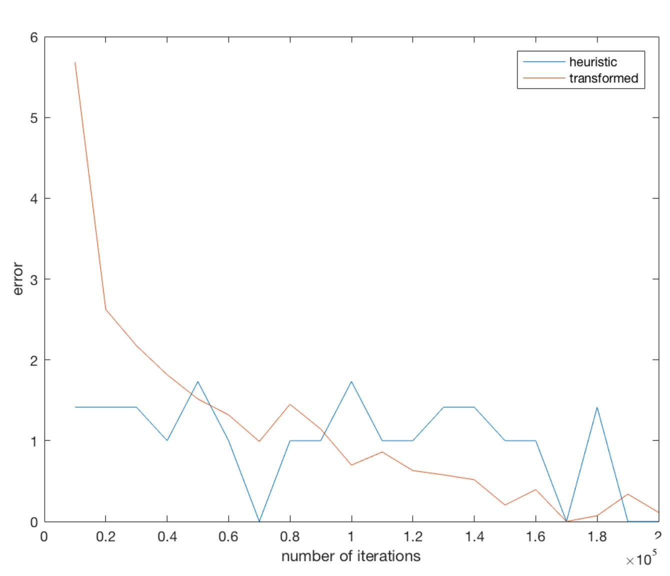

Figure 1. Performance Comparison with Heuristic[1]

*Due to the noises, the overbooking limits are averaged every 20,000 iterations

Reference

[1] Karaesmen, I., & Van Ryzin, G. (2004). Overbooking with substitutable inventory classes. *Operations Research*, *52*(1), 83-104.

[2] Chen, X., Gao, X., & Pang, Z. (2015). Preservation of structural properties in optimization with decisions truncated by random variables and its applications. *preprint*.

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